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Discrete Modeling of the Worm Spread with Random Scanning*

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SUMMARY In this paper, we derive a set of discrete time difference equations that models the spreading process of computer worms such as Code-Red and Slammer, which uses a common strategy called "random scanning" to spread through the Internet. We show that the derived set of discrete time difference equations has an exact relationship with the Kermack and McKendrick susceptible-infectious-removed (SIR) model, which is known as a standard continuous time model for worm spreading.

key words: *worm spread, discrete model, SIR model*

1. Introduction

The easy access and wide usage of the Internet makes it a primary target for malicious activities. The threat of these computer worms has become not only a major Internet security problem but a serious social and economic problem [2], [3]. Therefore, to defend against future worms, it is important to accurately understand the process of worm spread.

A family of worm spreading models has been proposed [2]–[4] based on the Kermack and McKendrick susceptible-infectious-removed (SIR) model [5], which is a continuous time epidemic model defined as

$$\frac{dS(t)}{dt} = -\alpha S(t)I(t), \tag{1}$$
$$\frac{dI(t)}{dt} = \alpha S(t)I(t) - \gamma I(t), \tag{2}$$
$$\frac{dR(t)}{dt} = \gamma I(t), \tag{3}$$

where α and γ are positive constants and $S(t)$, $I(t)$ and $R(t)$ represent the number of susceptible, infectious and removed hosts at time t, respectively. The SIR model is rather useful for explaining the actual data on the spread of Code-Red, Slammer and their evolutionary variants [2]–[4]. Therefore, it is important to obtain a solution of the SIR model to promote understanding of worm spread phenomena. However, it is difficult to find an exact solution of the SIR model. Although an explicit solution of the SIR model has been derived based on a homotopy analysis method [6], the solution is not tractable because it is given by a convergent series expression. A standard approach of avoiding the above issue is to consider a numerical solution of the SIR model, which can be obtained by discretization. However, a simple discretization generally causes a larger approximation error as the discretization interval becomes larger. Evidence for understanding this is that a logistic map, which is given by a simple discretization of Eqs. (1), (2), and (3) with $\gamma = 0$, shows chaotic behavior under some values of α. However, little attention has been given on constructing discrete time models, which are not simple reduction from continuous models by discretization and can be applied to various discretization intervals, to explain the process of worm spread, despite the fact that it is naturally discrete and the measurements related to its statistics take place, for practical reasons, at fixed intervals. Although Chen et al. proposed a discrete time model called the Analytical Active Worm Propagation (AAWP) model, which is the most relevant to this paper, for considering the effects of patching and disconnecting, they did not discuss the relationship between their model and the SIR model [7].

Against this background, in this paper, we first derive a set of discrete time difference equations that represents the characteristics of the spread of worms such as Code-Red and Slammer, which uses a common strategy called "random scanning" to spread through the Internet [2], [3]. In addition, we show that the derived set of discrete time difference equations has an exact relationship with the SIR model, while the AAWP model does not.

2. Scenario of Worm Spread

In random scanning, each infectious host independently generates random IP addresses and scans (attacks) the hosts that have those addresses. If the attacked hosts are susceptible (vulnerable), they become infectious. Each infectious host repeats this procedure until it is patched by countermeasures or disconnected from the network.

Let us define the following parameters (see Fig. 1):

- Let T (> 1) denote the number of hosts that are targets

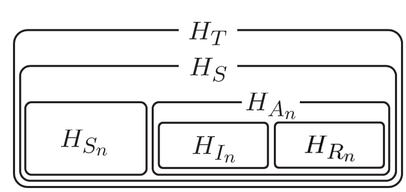

Fig. 1 Relationship among parameters.

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of attacks and let H_T denote the set of these hosts.

- Let S $(\leq T)$ denote the number of susceptible hosts before the worm spread begins and let H_S $(\subset H_T)$ denote the set of these hosts. Therefore, a host in $H_T \setminus H_S$ $(\subset H_T)$ is by nature insusceptible.
- Let S_n $(\leq S)$ denote the number of susceptible hosts at time tick n and let H_{S_n} $(\subset H_S)$ denote the set of these hosts, where S_n is a random variable.
- Let A_n $(\leq S)$ denote the total number of susceptible hosts that have been attacked by time tick n and let H_{A_n} $(\subset H_S)$ denote the set of these hosts, where A_n is a random variable. Thus, every host in H_{A_n} has been infected by time tick n. Note that $S_0 = S - A_0$.
- Let I_n $(\leq A_n)$ denote the total number of infectious hosts at time tick n and let H_{I_n} $(\subset H_{A_n})$ denote the set of these hosts, where I_n is a random variable. Thus, every host in H_{I_n} has been infected by time tick n, but none have been patched or disconnected by time tick n. In other words, every host in $H_{R_n} = H_{A_n} \setminus H_{I_n}$ $(\subset H_{A_n})$ has been infected by time tick n and has been removed by either patching or disconnecting by time tick n, where $R_n = A_n - I_n$ is a random variable. Note that $A_0 = I_0$.

The scenario of worm spread is described as follows. Although this scenario is the same as the one discussed in [7], we describe it again in detail to provide an exact analysis of worm spread with random scanning. Henceforth, $\mathbb{E}[X]$ denotes the expectation of X, and $\mathbb{E}[X|Y]$ denotes the conditional expectation of X given Y for random variables X and Y.

- At time tick $n + 1$, each host in H_{I_n} independently repeats a procedure that randomly selects a host in H_T and attacks it, where the number of repeats is s (≥ 1). If the attacked host is susceptible (i.e., the attacked host is in H_{S_n}), it becomes infectious. Note that the total number of attacks at time tick $n + 1$ is sI_n.
- Let $\bar{A}_{n+1,i}$ denote the total number of susceptible hosts attacked by the end of the i-th attack at time tick $n + 1$ and $\tilde{A}_{n+1,i}$ $(\in \{0, 1\})$ denote the number of susceptible hosts newly attacked at the i-th attack at time tick $n+1$, where $\bar{A}_{n+1,i}$ and $\tilde{A}_{n+1,i}$ are random variables. Here,

$$\bar{A}_{n+1,i} = A_n + \sum_{j=0}^{i} \tilde{A}_{n+1,j} = \bar{A}_{n+1,i-1} + \tilde{A}_{n+1,i} \qquad (4)$$

  holds, where $\bar{A}_{n+1,sI_n} = A_{n+1}$, $\bar{A}_{n+1,0} = A_n$, and $\tilde{A}_{n+1,0} = 0$ are defined for the convenience of notation.
- At time tick $n + 1$, each host in $H_{S_n} \setminus (H_{A_{n+1}} \setminus H_{A_n})$ (i.e., each of the susceptible hosts that has never been attacked by time tick $n + 1$) will be patched at rate p. This setting gives us

$$\begin{aligned}&\mathbb{E}[S_{n+1}|A_{n+1}, A_n, S_n] \\ &\quad = (1 - p)\{S_n - (A_{n+1} - A_n)\}. \end{aligned} \qquad (5)$$

  Then, at time tick $n + 1$, each host in $H_{A_{n+1}} \setminus H_{A_n}$ (i.e., each of the newly attacked hosts at time tick $n+1$) will be patched at rate p. Moreover, at time tick $n + 1$, each host in H_{I_n} (i.e., each of the infectious hosts at time tick n) will be either patched or disconnected at a certain rate after the sI_n attacks, where the patching and disconnecting rates are denoted by p and d, respectively (note: the rate of either patching or disconnecting is $p + d$, where $0 \leq p + d < 1$ and $0 \leq p, d < 1$). The above settings give us

$$\begin{aligned}&\mathbb{E}[I_{n+1}|A_{n+1}, A_n, I_n] \\ &\quad = (1 - p - d)I_n + (1 - p)(A_{n+1} - A_n). \end{aligned} \qquad (6)$$

  Equations (5) and (6) mean that all the unpatched hosts at time tick n (note: their number is $S_n + I_n$, where $S_n + I_n = \{S_n - (A_{n+1} - A_n)\} + \{I_n + (A_{n+1} - A_n)\}$) are patched after the sI_n attacks at the same rate p, and all the infectious hosts at time tick n are disconnected after the sI_n attacks at rate d. Note that all hosts that are patched (i.e., immunized) or disconnected (i.e., quarantined) will never be infectious, even if they are attacked.

3. Analysis

3.1 Relationship with SIR Model

We first present the following theorem, which describes the process of worm spread with random scanning. The proof of this theorem is given in Appendix A.

Theorem 1: For S_n, I_n, and R_n $(n = 0, 1, \ldots)$,

$$\mathbb{E}[S_{n+1}] = (1 - p)\mathbb{E}[S_n]\left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}, \qquad (7)$$

$$\begin{aligned}\mathbb{E}[I_{n+1}] &= (1 - p - d)\mathbb{E}[I_n] \\ &\quad + (1 - p)\mathbb{E}[S_n]\left\{1 - \left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}\right\}, \end{aligned} \qquad (8)$$

$$\begin{aligned}\mathbb{E}[R_{n+1}] &= \mathbb{E}[R_n] + (p + d)\mathbb{E}[I_n] \\ &\quad + p\mathbb{E}[S_n]\left\{1 - \left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}\right\} \end{aligned} \qquad (9)$$

hold by assuming that

$$\mathbb{E}[A_{n+1}] = \mathbb{E}[A_{n+1}|I_n = \mathbb{E}[I_n]], \qquad (10)$$

$$\mathbb{E}[A_n] = \mathbb{E}[A_n|I_n = \mathbb{E}[I_n]], \qquad (11)$$

$$\mathbb{E}[S_n] = \mathbb{E}[S_n|I_n = \mathbb{E}[I_n]], \qquad (12)$$

hold[†].

We next present the following theorem, which describes the exact relationship between the derived set of discrete time difference equations and the SIR model. We can obtain the following theorem by setting $n = \frac{t}{\delta}$, $s = \bar{s}\delta$,

[†]Although Eqs. (10), (11) and (12) are not satisfied in general, we assume these equations hold for convenience of computation in the following discussion.

$p = \bar{p}\delta$, and $d = \bar{d}\delta$ in Theorem 1, where $\delta \downarrow 0$, and we denote $S(\delta n) = S_n$, $I(\delta n) = I_n$, and $R(\delta n) = R_n$ $(n = 0, 1, \ldots)$. The proof of this theorem is given in Appendix B.

Theorem 2: Assuming Eqs. (10), (11), and (12),

$$\frac{d\mathbb{E}[S(t)]}{dt} = -\bar{a}\mathbb{E}[S(t)]\mathbb{E}[I(t)] - \bar{p}\mathbb{E}[S(t)], \tag{13}$$

$$\frac{d\mathbb{E}[I(t)]}{dt} = \bar{a}\mathbb{E}[S(t)]\mathbb{E}[I(t)] - (\bar{p} + \bar{d})\mathbb{E}[I(t)], \tag{14}$$

$$\frac{d\mathbb{E}[R(t)]}{dt} = (\bar{p} + \bar{d})\mathbb{E}[I(t)], \tag{15}$$

hold for $S(t)$, $I(t)$, and $R(t)$, where $\bar{p} \geq 0$, $\bar{d} \geq 0$, and

$$\bar{a} = -\left\{\ln\left(1 - \frac{1}{T}\right)^{\bar{s}}\right\} > 0. \tag{16}$$

Note that Eqs. (13), (14), and (15) are equivalent to the SIR model given by Eqs. (1), (2), and (3) when $\bar{p} = 0$. That is, the SIR model can be obtained as a special case of the proposed discrete time model (i.e., Eqs. (7), (8), and (9)) when the discretization interval, δ, is small (i.e., $\delta \downarrow 0$). On the other hand, the proposed discrete time model is derived without assuming that the discretization interval is small. This indicates that the proposed discrete-time model is favorable for explaining the worm spread phenomena with various time-resolutions. Therefore, the proposed discrete time model can be used for explaining a wider variety of the worm spread phenomena than with the SIR model (and its simple discretization). In this sense, the proposed discrete time model can be regarded as a generalization of the SIR model. In addition, note that Eq. (16) reveals a non-trivial feature in which the attack ratio, $\bar{a}$, is proportional to the logarithm of $1 - 1/T$.

3.2 Relationship with AAWP Model

If $p = 0$ and $d = 0$, using Eqs. (7) and (8), we easily obtain

$$\mathbb{E}[I_{n+1}] = \mathbb{E}[I_n] + \{S - \mathbb{E}[I_n]\}\left\{1 - \left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}\right\}, \tag{17}$$

which is the same difference equation as the one proposed in [7]. On the other hand, for $p \neq 0$ or $d \neq 0$, the following equation was proposed in [7], implying the same scenario as described in Sect. 2.

$$\begin{aligned}\mathbb{E}[I_{n+1}] =&(1 - p - d)\mathbb{E}[I_n] \\ &+ \{S(1 - p)^n - \mathbb{E}[I_n]\}\left\{1 - \left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}\right\}.\end{aligned} \tag{18}$$

In [7], Eq. (18) was derived by simply replacing $\mathbb{E}[I_n]$ (in the first term of the right-hand side) and S in Eq. (17) by $(1-p-d)\mathbb{E}[I_n]$ and $S(1-p)^n$, respectively. The replacement was intuitively justified based on the following reasons: (i) there will be $(p + d)\mathbb{E}[I_n]$ infected hosts that will change to patched or disconnected hosts at time tick $n + 1$, and (ii) the number of susceptible hosts, which can be infected by attacks at time tick $n + 1$, is given by the difference between the number of hosts that have never been patched by time tick $n+1$, $S(1-p)^n$, and the number of infected hosts at time tick n, $\mathbb{E}[I_n]$. However, it is important to note that argument (ii) is not correct because it overlooks the fact that the hosts that have never been patched by time tick $n + 1$, $S(1 - p)^n$, includes those that have been infected and then disconnected by time tick $n+1$, which cannot be infected by attacks at time tick $n + 1$ (because they have already been disconnected). Therefore, the AAWP model overestimates the number of hosts that can be infected by attacks at time tick $n + 1$. In fact, we can show that

$$\text{Eq. (8)} \leq \text{Eq. (18)} \tag{19}$$

holds, where the equality holds if and only if $p = 0$ and $d = 0$. The proof of this equality is given in Appendix C. This inequality means that the AAWP model underestimates the effects of patching and disconnecting (i.e., over-estimates the number of infectious nodes) compared with the model derived in this paper, except when $p = 0$ and $d = 0$. The above observation indicates that the AAWP model does not have an exact relationship with the SIR model.

3.3 Simulation

In this section, we provide some simulation results in order to see the property of the proposed and AAWP models, where the scenario of worm spread used in this simulation is strict agreement with the model described in Sect. 2. Figure 2 shows the transition of the (average) number of infectious hosts I_n with various parameter settings given by the proposed model (Eq. (14): solid line), AAWP model (Eq. (18): dotted line), and corresponding simulation (cross), where we used an average of 100 simulations. The horizontal axis of Fig. 2 denotes the value of time tick n. Although the simulation results are given at all time ticks, those every 100 time ticks are depicted as cross symbols. Similar results are also obtained even when the other values of parameters are used.

As shown in this figure, (i) both the proposed model and the AAWP model can well explain the worm spread when $p = 0$ and $d = 0$ and (ii) the proposed model can explain worm spread better than the AAWP model when $p \neq 0$ and $d \neq 0$. The first point was already observed in [7]. Note that the proposed model is exactly the same as the AAWP model when $p = 0$ and $d = 0$ (see Sect. 3.2). On the other hand, the second point reveals that the AAWP model is not valid when $p \neq 0$ and $d \neq 0$. Note that the relationship between the proposed model and the AAWP model shown in this simulation corresponds with that of inequality (19).

4. Conclusion

We derived a discrete time model for worm spreading with random scanning. In addition, we showed that the derived model has an exact relationship with a standard continuous time model known as the Kermack and McKendrick SIR

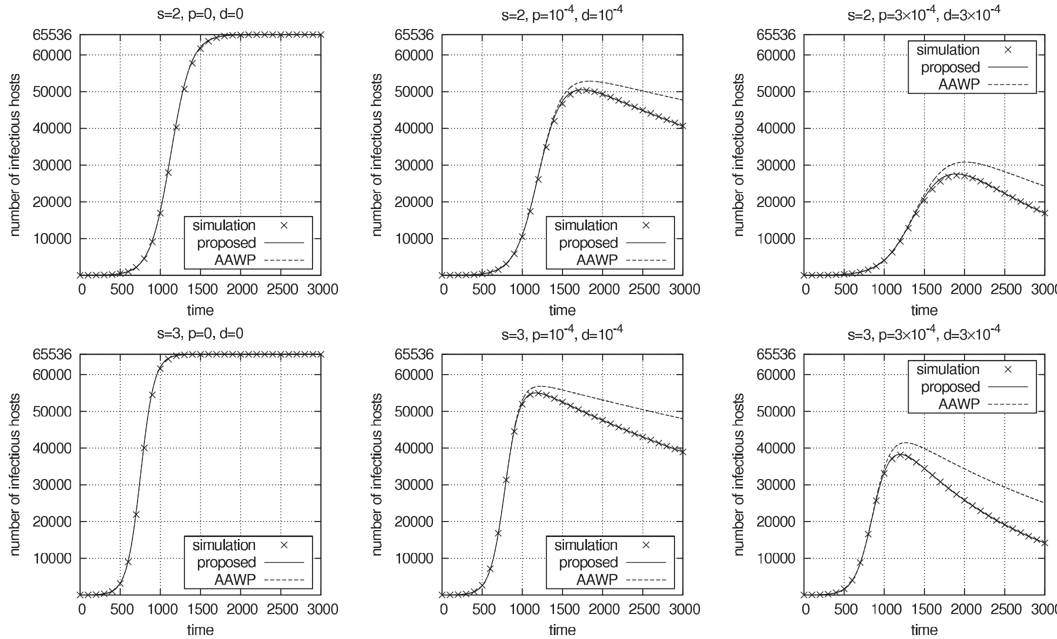

Fig. 2 Transition of the number of infectious hosts. All cases are for $T = 2^{24}$, $S = 2^{16}$, $I_0 = 10$.

model. Since the SIR model has obtained the status as an origin of various existing continuous time models for worm spread, we can expect that the derived model has potential for providing a basis for discrete modeling of worm spread.

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Appendix A: Proof of Theorem 1

Under the scenario described in Sect. 2, we obtain

$$\mathbb{E}[\tilde{A}_{n+1,i}|A_n, \bar{A}_{n+1,i-1}, S_n, I_n] = \frac{1}{T}\{S_n - (\bar{A}_{n+1,i-1} - A_n)\}, \quad \text{(A· 1)}$$

for $i = 0,\ 1,\ \ldots,\ sI_n$. Using Eq. (A· 1), we obtain

$$\mathbb{E}[\tilde{A}_{n+1,i}|I_n] = \frac{1}{T}\{\mathbb{E}[S_n|I_n] - (\mathbb{E}[\bar{A}_{n+1,i-1}|I_n] - \mathbb{E}[A_n|I_n])\}. \quad \text{(A· 2)}$$

In addition, using Eqs. (4) and (A· 2), we obtain

$$\begin{aligned}
&\mathbb{E}[\bar{A}_{n+1,i}|I_n]\\
&\quad = \mathbb{E}[\bar{A}_{n+1,i-1}|I_n]\\
&\qquad + \frac{1}{T}\{\mathbb{E}[S_n|I_n] - (\mathbb{E}[\bar{A}_{n+1,i-1}|I_n] - \mathbb{E}[A_n|I_n])\}\\
&\quad = \mathbb{E}[S_n|I_n] + \mathbb{E}[A_n|I_n]\\
&\qquad - \{\mathbb{E}[S_n|I_n] - (\mathbb{E}[\bar{A}_{n+1,i-1}|I_n] - \mathbb{E}[A_n|I_n])\}\left(1 - \frac{1}{T}\right).
\end{aligned} \quad \text{(A· 3)}$$

Assuming that $\bar{A}_{n+1,sI_n} = A_{n+1}$ and using Eq. (A· 3) with $i = sI_n$, we obtain

$$\begin{aligned}
&\mathbb{E}[A_{n+1}]\\
&\quad = \mathbb{E}[\bar{A}_{n+1,sI_n}]\\
&\quad = \mathbb{E}[\bar{A}_{n+1,sI_n}|I_n]\\
&\quad = \mathbb{E}[S_n|I_n] + \mathbb{E}[A_n|I_n]\\
&\qquad - \{\mathbb{E}[S_n|I_n] - (\mathbb{E}[\bar{A}_{n+1,sI_n-1}|I_n] - \mathbb{E}[A_n|I_n])\}\left(1 - \frac{1}{T}\right).
\end{aligned} \quad \text{(A· 4)}$$

Then, by substituting Eq. (A· 3) with $i = sI_n - 1$ into Eq. (A· 4), we obtain

$$\begin{aligned}&\mathbb{E}[A_{n+1}]\\ &= \mathbb{E}[S_n|I_n] + \mathbb{E}[A_n|I_n]\\ &\quad - \{\mathbb{E}[S_n|I_n] - (\mathbb{E}[\bar{A}_{n+1,sI_n-2}|I_n] - \mathbb{E}[A_n|I_n])\}\left(1 - \frac{1}{T}\right)^2.\end{aligned} \tag{A· 5}$$

Therefore, repeating the above calculation and assuming Eqs. (10), (11), and (12), we obtain

$$\mathbb{E}[A_{n+1}] = \mathbb{E}[S_n] + \mathbb{E}[A_n] - \mathbb{E}[S_n]\left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}, \tag{A· 6}$$

where we used $\bar{A}_{n+1,0} = A_n$.

Using Eq. (A· 6), Eqs. (7) and (8) can be directly derived from Eqs. (5) and (6), respectively. In addition, using Eqs. (8) and (A· 6) and noting that $R_n = A_n - I_n$, Eq. (9) can be derived.

Appendix B: Proof of Theorem 2

Using Eqs. (7), (8), and (9), we obtain

$$\begin{aligned}&\mathbb{E}[S(t+\delta)] - \mathbb{E}[S(t)]\\ &\quad = -(1 - \bar{p}\delta)\mathbb{E}[S(t)]\left\{1 - \left(1 - \frac{1}{T}\right)^{\bar{s}\delta\mathbb{E}[I(t)]}\right\}\\ &\qquad - \bar{p}\delta\mathbb{E}[S(t)],\end{aligned} \tag{A· 7}$$

$$\begin{aligned}&\mathbb{E}[I(t+\delta)] - \mathbb{E}[I(t)]\\ &\quad = (1 - \bar{p}\delta)\mathbb{E}[S(t)]\left\{1 - \left(1 - \frac{1}{T}\right)^{\bar{s}\delta\mathbb{E}[I(t)]}\right\}\\ &\qquad - (\bar{p}\delta + \bar{d}\delta)\mathbb{E}[I(t)],\end{aligned} \tag{A· 8}$$

$$\begin{aligned}&\mathbb{E}[R(t+\delta)] - \mathbb{E}[R(t)]\\ &\quad = \bar{p}\delta\mathbb{E}[S(t)]\left\{1 - \left(1 - \frac{1}{T}\right)^{\bar{s}\delta\mathbb{E}[I(t)]}\right\}\\ &\qquad + (\bar{p}\delta + \bar{d}\delta)\mathbb{E}[I(t)].\end{aligned} \tag{A· 9}$$

Dividing Eqs. (A· 7), (A· 8), and (A· 9) by δ and taking the limit as $\delta \downarrow 0$, we can obtain Eqs. (13), (14), and (15), respectively, where we used

$$\lim_{\delta\downarrow 0} \frac{1 - \left(1 - \frac{1}{T}\right)^{\bar{s}\delta\mathbb{E}[I(t)]}}{\delta} = -\left\{\ln\left(1 - \frac{1}{T}\right)^{\bar{s}}\right\}\mathbb{E}[I(t)].$$

Appendix C: Proof of Inequality (19)

Using Eq. (8), we obtain

$$\mathbb{E}[I_{n+1}] \le (1 - p - d)\mathbb{E}[I_n] + \mathbb{E}[S_n]\left\{1 - \left(1 - \frac{1}{T}\right)^{s\mathbb{E}[I_n]}\right\}, \tag{A· 10}$$

where the equality holds if and only if $p = 0$. In addition, using Eqs. (8) and (A· 6), we obtain

$$\begin{aligned}\mathbb{E}[I_n] \le&(1 - p)^n\mathbb{E}[I_0]\\ &+ \sum_{i=1}^{n}(1 - p)^{n-i+1}(\mathbb{E}[A_i] - \mathbb{E}[A_{i-1}]),\end{aligned} \tag{A· 11}$$

where the equality holds if and only if $d = 0$. Moreover, using Eqs. (7) and (A· 6), we obtain

$$\begin{aligned}\mathbb{E}[S_n] =&(1 - p)^n S - (1 - p)^n\mathbb{E}[I_0]\\ &- \sum_{i=1}^{n}(1 - p)^{n-i+1}(\mathbb{E}[A_i] - \mathbb{E}[A_{i-1}])\\ \le&(1 - p)^n S - \mathbb{E}[I_n],\end{aligned} \tag{A· 12}$$

where the last inequality is given by Eq. (A· 11). Substituting Eq. (A· 12) into (A· 10), we obtain Eq. (19).